Some results on change-point detection in cross-sectional dependence of multivariate data with changes in marginal distributions.

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Abstract

Tests for break points detection in the law of random vectors have been proposed in several papers. Nevertheless, they have often little power for alternatives involving a change in the dependence between components of vectors. Specific tests for detection of a change in the copula of random vectors have also been proposed in recent papers, but they do not allow to conclude of a change in the dependence structure without condition that the margins are constant. The goal of this article is to propose a test for detection of a break in the copula when changes in marginal distribution occurs at known instants. The performances of this test are illustrated by Monte Carlo simulations.

Keywords: non-parametric tests, sequential empirical copula process, Monte Carlo experiments

1. Introduction

Let $\boldsymbol{X}$ be a d-dimensional random vector ($d \geq 2$), with cumulative distribution function (c.d.f.) F and marginal cumulative distribution functions (m.c.d.f.s) $F_1, \dots, F_d$. When the m.c.d.f.s $F_1, \dots, F_d$ are continuous, Sklar's Theorem (see Sklar, 1959) allows us to say that there exists a unique function C called copula, characterizing the dependence of random vector $\boldsymbol{X}$, such that F can be written as:

$$F(\boldsymbol{x}) = C(F_1(x_1), \dots, F_d(X_d)), \qquad \boldsymbol{x} \in \mathbb{R}^d. \tag{1}$$

Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be d-dimensional observations. The purpose of change-points detection is to test the hypothesis

$$H_0 : \exists F \text{ such as } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ have c.d.f. } F, \tag{2}$$

against $\neg H_0$. The equation (1) involves that H_0 can be rewritten as $H_0 = H_{0,m} \cap H_{0,c}$, with

$$H_{0,m} : \exists F_1, \dots F_d \text{ such as } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ have m.c.d.f. } F_1, \dots F_d, \tag{3}$$

$$H_{0,c} : \exists C \text{ such as } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ have copula } C. \tag{4}$$

A change either in the copula of random vectors or in one of m.c.d.f.s implies the rejection of the null hypothesis H_0. Many non-parametric tests for H_0 based on empirical processes are present in the literature; see for example Bai (1994); Csörgő and Horváth (1997); Inoue (2001). These tests are not very sensitive to detect a change in the copula which leaves the m.c.d.f.s unchanged. This conclusion is highlighted in Holmes et al. (2013, Section 2) through Monte Carlo simulations.

Non-parametric tests for break detection, sensitive to changes in the copula of observations and based on the two-sided sequential empirical copula process are considered in Bücher et al. (2014). These tests do not allow to conclude in favour of $\neg H_{0,c}$ without condition on the constancy of m.c.d.f.s. In many situations, see for example section 5.2, a specific event can lead to changes in the marginal cumulative distributions. The question then becomes whether the specific event changes the copula or not. The aim of this paper is to propose a test to detect a change in the dependence structure of random vectors, sensitive to changes in copula of observations and adapted in the case of alternative hypotheses involving abrupt changes in m.c.d.f.s.

This paper is organized as follows. The procedure to test the null hypothesis of a break in c.d.f. when a change in the m.c.d.f.s occurs is presented in Section 2. An adaptation of results of Section 2 when multiple changes in m.c.d.f.s occur is described in Section 3. Section 4 contains the results of Monte Carlo simulations. Finally, Section 5 reports brief discussions about the case of *alpha*-mixing observations and presents an illustration on a specific situation.

2. Break detection in the copula when a break time in the m.c.d.f.s is known

In the sequel, the weak convergence, denoted by $\rightsquigarrow$, must be understood as being the weak convergence in the sense of Definition 1.3.3 in van der Vaart and Wellner (2000). For a set T, $\ell^\infty(T)$ denotes the space of bounded real-valued functions on T equipped with the uniform metric.

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be d-dimensional random vectors ($d \geq 2$) and consider for $1 \leq k \leq l \leq n$ the empirical copula $C_{k:l}$ of the sub-sample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ as suggested in Deheuvels (1979):

$$C_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \prod_{j=1}^{d} \mathbf{1}(F_{k:l,j}(X_{ij}) \leq u_j), \qquad \boldsymbol{u} \in [0,1]^d, \tag{5}$$

where for $j = 1, \ldots, d$, $F_{k:l,j}$ is the empirical cumulative distribution function (e.c.d.f.) of sample $X_{kj}, \ldots, X_{lj}$:

$$F_{k:l,j}(x) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(X_{ij} \leq x), \qquad x \in \mathbb{R}. \tag{6}$$

In Bücher et al. (2014), the following Cramér–von Mises's type statistic to test $\neg H_0$ is suggested:

$$S_n = \sup_{s \in [0,1]} \sqrt{n} \lambda_n(s,1) \lambda_n(0,s) \int_{[0,1]^d} \{C_{1:\lfloor ns \rfloor}(\boldsymbol{u}) - C_{\lfloor ns \rfloor + 1:n}(\boldsymbol{u})\}^2 \mathrm{d}C_{1:n}(\boldsymbol{u}), \tag{7}$$

where $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$, $s \leq t \in [0,1]$.

Monte Carlo simulations (see section 5 of Bücher et al., 2014) highlighted that a strategy of boostraping with independent or dependent multipliers according to the observations (see Bücher and Kojadinovic, 2015; Bücher et al., 2014) of the statistic S_n leads to very good performances in term of powers for alternatives hypotheses that involve a change in copula which leave the m.c.d.f.s unchanged.

Let us suppose that it exists a break time $m = \lfloor nb \rfloor$ in m.c.d.f.s, $b \in (0,1)$ known. We propose a test for $H_0^m = H_{0,c} \cap H_{1,m}$, where $H_{0,c}$ is defined in (4) and $H_{1,m}$ is defined by:

$$H_{1,m} : \exists F_1, \ldots F_d \text{ and } F'_1, \ldots F'_d \text{ such that } \begin{array}{c} \boldsymbol{X}_1, \ldots, \boldsymbol{X}_m \text{ have m.c.d.f. } F_1, \ldots F_d, \\ \boldsymbol{X}_{m+1}, \ldots, \boldsymbol{X}_n \text{ have m.c.d.f. } F'_1, \ldots F'_d. \end{array} \tag{8}$$

Note that we do not suppose that $F'_1, \ldots F'_d$ are necessarily different from $F_1, \ldots F_d$. In other words, we do not assume a change in the m.c.d.f.s. However we suppose that if there is a change in the m.c.d.f.s, it is a unique and abrupt change at time m.

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be d-dimensional random vectors with unknown copula C, such that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ have m.c.d.f.s $F_1, \ldots, F_d$ and $\boldsymbol{X}_{m+1}, \ldots, \boldsymbol{X}_n$ have m.c.d.f.s $F'_1, \ldots, F'_d$, where $F_1, \ldots, F_d, F'_1, \ldots, F'_d$ are unknown and the break time $m = \lfloor nb \rfloor$, $b \in (0,1)$ is known.

For $i \in \{1, \ldots, n\}$, let us consider the random vectors $\boldsymbol{U}_{i,m}$ defined by

$$\boldsymbol{U}_{i,m} = \begin{cases} (F_1(X_{i1}), \ldots, F_d(X_{id})) & i \in \{1, \ldots, m\} \\ (F'_1(X_{i1}), \ldots, F'_d(X_{id})) & i \in \{m+1, \ldots, n\}. \end{cases} \tag{9}$$

Note that the vectors $\boldsymbol{U}_{i,m}$, $i = 1, \ldots, n$ have C for c.d.f. For $i \in \{1, \ldots, n\}$, let $\hat{\boldsymbol{U}}_{i,m}^{1:n}$ defined by:

$$\hat{\boldsymbol{U}}_{i,m}^{1:n} = \begin{cases} (F_{1:m,1}(X_{i1}), \ldots, F_{1:m,d}(X_{id})) & i \in \{1, \ldots, m\} \\ (F_{m+1:n,1}(X_{i1}), \ldots, F_{m+1:n,d}(X_{id})) & i \in \{m+1, \ldots, n\}, \end{cases}$$

where for all $1 \leq k \leq l \leq n$ and $j = 1, \ldots, d$ $F_{k:l,j}$ is the empirical c.d.f. of $X_{kj}, \ldots, X_{lj}$ as defined in (6). The vectors $\hat{\boldsymbol{U}}_{i,m}^{1:n}$, $i = 1, \ldots, n$ can be seen as pseudo-observations of copula C. An estimator of C is given by the empirical distribution of $\hat{\boldsymbol{U}}_{1,m}^{1:n}, \ldots, \hat{\boldsymbol{U}}_{n,m}^{1:n}$:

$$C_{1:n,m}(\boldsymbol{u}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(\hat{\boldsymbol{U}}_{i,m}^{1:n} \leq \boldsymbol{u}), \quad \boldsymbol{u} \in [0,1]^d.$$

This estimator can be rewritten for $\boldsymbol{u} \in [0,1]^d$ by:

$$C_{1:n,m}(\boldsymbol{u}) = \frac{m}{n} C_{1:m}(\boldsymbol{u}) + \frac{n-m}{n} C_{m+1:n}(\boldsymbol{u}),$$

where for any subsample $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$, $1 \leq k \leq l \leq n$, $C_{k:l}$ is the empirical c.d.f. of random vectors $\hat{\boldsymbol{U}}_k^{k:l}, \dots, \hat{\boldsymbol{U}}_l^{k:l}$ defined in (5), and for $i = k, \dots, l$

$$\hat{\boldsymbol{U}}_i^{k:l} = (F_{k:l,1}(X_{i1}), \dots, F_{k:l,d}(X_{i1})).$$

More particularly, for $1 \leq k \leq l \leq n$, $m \in \{1, \dots, n-1\}$ and $\boldsymbol{u} \in [0,1]^d$,

$$C_{k:l,m}(\boldsymbol{u}) = \begin{cases} \frac{m-k+1}{l-k+1} C_{k:m}(\boldsymbol{u}) + \frac{l-m}{l-k+1} C_{m+1:l}(\boldsymbol{u}) & m \in [k,l], \\ C_{k:l}(\boldsymbol{u}) & m \notin [k,l]. \end{cases} \tag{10}$$

For a subsample $\boldsymbol{X}_k \dots, \boldsymbol{X}_l$, $1 \leq k \leq l \leq n$ consider the following pseudo-observations of copula C:

$$\hat{\boldsymbol{U}}_{i,m}^{k:l} = \begin{cases} \begin{cases} (F_{k:m,1}(X_{i1}), \dots, F_{k:m,d}(X_{i1})) & i \in \{k, \dots, m\} \\ (F_{m+1:l,1}(X_{i1}), \dots, F_{m+1:l,d}(X_{i1})) & i \in \{m+1, \dots, l\} \end{cases} & m \in [k,l], \\ (F_{k:l,1}(X_{i1}), \dots, F_{k:l,d}(X_{i1})) & m \notin [k,l], \end{cases}$$

$F_{k:l}$ defined in (6). Then $C_{k:l,m}$ defined in (10) is the empirical cumulative distribution of $\hat{\boldsymbol{U}}_{k,m}^{k:l}, \dots, \hat{\boldsymbol{U}}_{l,m}^{k:l}$.

The corresponding two-sided sequential empirical copula process is defined by

$$\begin{aligned} \mathbb{C}_{n,m}(s,t,\boldsymbol{u}) &= \sqrt{n}\lambda_n(s,t)\{C_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,m}(\boldsymbol{u}) - C(\boldsymbol{u})\}, \quad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d, \\ &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \{\mathbf{1}(\hat{\boldsymbol{U}}_{i,m}^{\lfloor ns \rfloor+1:\lfloor nt \rfloor} \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \end{aligned} \tag{11}$$

$\Delta = \{(s,t) \in [0,1]^2 | s \leq t\}$. The test statistic proposed in this paper is based on the process $\mathbb{D}_{n,m}$, defined by

$$\mathbb{D}_{n,m}(s,\boldsymbol{u}) = \sqrt{n}\lambda_n(0,s)\lambda_n(s,1)\{C_{1:\lfloor ns \rfloor,m}(\boldsymbol{u}) - C_{\lfloor ns \rfloor+1:n,m}(\boldsymbol{u})\}, \quad (s,\boldsymbol{u}) \in [0,1]^{d+1}.$$

Note that $\mathbb{D}_{n,m}$ can be rewritten as

$$\mathbb{D}_{n,m}(s,\boldsymbol{u}) = \lambda_n(s,1)\mathbb{C}_{n,m}(0,s,\boldsymbol{u}) - \lambda_n(0,s)\mathbb{C}_{n,m}(s,1,\boldsymbol{u}), \qquad (s,\boldsymbol{u}) \in [0,1]^{d+1}.$$

Similarly to S_n defined in (7), we consider the Cramér–von Mises statistic

$$\begin{aligned} S_{n,m} &= \sup_{s \in [0,1]} \int_{[0,1]^d} \{\mathbb{D}_{n,m}(s,\boldsymbol{u})\}^2 \d C_{1:n,m}(\boldsymbol{u}) \\ &= \max_{1 \leq k \leq n-1} \frac{1}{n} \sum_{i=1}^{n} \{\mathbb{D}_{n,m}(k/n, \hat{\boldsymbol{U}}_{i,m}^{1:n})\}^2. \end{aligned} \tag{12}$$

The asymptotic behaviour of the empirical process $\mathbb{D}_{n,m}$ is given on Proposition 2.1, proved in Appendix A. The result is obtained under the following non-restrictive condition, proposed in Segers (2012):

Condition 2.1. *For any $j \in \{1, \dots, d\}$, the partial derivatives $\dot{C}_j = \partial C / \partial u_j$ exist and are continuous on $V_j = \{\boldsymbol{u} \in [0,1]^d, u_j \in (0,1)\}$.*

Proposition 2.1. *Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be d-dimensional independent random vectors with copula C, such that for $b \in (0,1)$ known and $m = \lfloor nb \rfloor$, the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_m$ have m.c.d.f.s $F_1, \dots, F_d$ and the random vectors $\boldsymbol{X}_{m+1}, \dots, \boldsymbol{X}_n$ have m.c.d.f.s $F_1', \dots, F_d'$.*

Then, under Condition 2.1, the process $\mathbb{D}_{n,m}$ converges weakly in $\ell^\infty([0,1]^{d+1})$, to a stochastic process $\mathbb{D}_C$ defined by

$$\mathbb{D}_C(s,\boldsymbol{u}) = \mathbb{C}_C^0(s,\boldsymbol{u}) - s\mathbb{C}_C^0(1,\boldsymbol{u}), \quad (s,\boldsymbol{u}) \in [0,1]^{d+1}, \tag{13}$$

where for $(s,\boldsymbol{u}) \in [0,1]^{d+1}$,

$$\mathbb{C}_C^0(s,\boldsymbol{u}) = \mathbb{Z}_C(s,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\mathbb{Z}_C(s,\boldsymbol{u}^{\{j\}}) \tag{14}$$

with $\mathbb{Z}_C$ is a tight centred Gaussian process with covariance function

$$\operatorname{cov}\{\mathbb{Z}_C(s,\boldsymbol{u}), \mathbb{Z}_C(t,\boldsymbol{v})\} = \min(s,t)\{C(\boldsymbol{u} \wedge \boldsymbol{v}) - C(\boldsymbol{u})C(\boldsymbol{v})\}, \quad (s,\boldsymbol{u}),(t,\boldsymbol{v}) \in [0,1]^{d+1},$$

$\boldsymbol{u} \wedge \boldsymbol{v} = (\min(u_1,v_1), \dots, \min(u_d,v_d))$ and $\boldsymbol{u}^{\{j\}} = (1, \dots, 1, u_j, 1, \dots, 1)$.

To resample $\mathbb{D}_{n,m}$, we note that for $b \le s$ and $\boldsymbol{u} \in [0,1]^d$, $\mathbb{D}_{n,m}$ rewrites as

$$\mathbb{D}_{n,m}(s,\boldsymbol{u}) = \lambda_n(s,1)\{\mathbb{C}_{n,m}(0,b,\boldsymbol{u}) + \mathbb{C}_{n,m}(b,s,\boldsymbol{u})\} - \lambda_n(0,s)\mathbb{C}_{n,m}(s,1,\boldsymbol{u})$$

and for $b \ge s$ and $\boldsymbol{u} \in [0,1]^d$ as

$$\mathbb{D}_{n,m}(s,\boldsymbol{u}) = \lambda_n(s,1)\mathbb{C}_{n,m}(0,s,\boldsymbol{u}) - \lambda_n(0,s)\{\mathbb{C}_{n,m}(s,b,\boldsymbol{u}) + \mathbb{C}_{n,m}(b,1,\boldsymbol{u})\}.$$

Let B a large integer and consider for $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$ and for $\beta = 1,\dots,B$, the processes

$$\check{\mathbb{B}}^{(\beta)}_{n,m}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_i^{(\beta)} \left\{ \mathbf{1}(\hat{\boldsymbol{U}}_{i,m}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le \boldsymbol{u}) - C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,m}(\boldsymbol{u}) \right\},$$

and

$$\check{\mathbb{C}}^{(\beta)}_{n,m}(s,t,\boldsymbol{u}) = \check{\mathbb{B}}^{(\beta)}_{n,m}(s,t,\boldsymbol{u}) - \frac{1}{\sqrt{n}} \sum_{j=1}^{d} \dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor,m}(\boldsymbol{u}) \check{\mathbb{B}}^{(\beta)}_{n,m}(s,t,\boldsymbol{u}^{\{j\}}), \tag{15}$$

with for $\beta = 1,\dots,B$, and $1 \le i \le n$, $\xi_i^{(\beta)}$ are i.i.d. standard normal random variables.

Re-sampling versions $\check{\mathbb{D}}^{(\beta)}_{n,m}$ of $\mathbb{D}_{n,m}$ can be obtained for $\beta = 1,\dots,B$ and $(s,\boldsymbol{u}) \in [b,1] \times [0,1]^d$ by

$$\check{\mathbb{D}}^{(\beta)}_{n,m}(s,\boldsymbol{u}) = \lambda_n(s,1)\{\check{\mathbb{C}}^{(\beta)}_{n,m}(0,b,\boldsymbol{u}) + \check{\mathbb{C}}^{(\beta)}_{n,m}(b,s,\boldsymbol{u})\} - \lambda_n(0,s)\check{\mathbb{C}}^{(\beta)}_{n,m}(s,1,\boldsymbol{u})$$

and for $(s,\boldsymbol{u}) \in [0,b] \times [0,1]^d$ by

$$\check{\mathbb{D}}^{(\beta)}_{n,m}(s,\boldsymbol{u}) = \lambda_n(s,1)\check{\mathbb{C}}^{(\beta)}_{n,m}(0,s,\boldsymbol{u}) - \lambda_n(0,s)\{\check{\mathbb{C}}^{(\beta)}_{n,m}(s,b,\boldsymbol{u}) + \check{\mathbb{C}}^{(\beta)}_{n,m}(b,1,\boldsymbol{u})\}.$$

For $j = 1,\dots,d$, the functions $\dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor,m}$ appearing in (15) are an adaptation of the estimator of $\dot{C}_j$ proposed in section 4.2 in Bücher et al. (2014) consisting in simple differencing at a bandwidth $h_{k:l} = \min\{(l-k+1)^{-1/2}, 1/2\}$ of the empirical copula process:

$$\dot{C}_{j,k:l,m}(\boldsymbol{u}) = \frac{C_{k:l,m}(\boldsymbol{u}^{j,+}) - C_{k:l,m}(\boldsymbol{u}^{j,-})}{u_j^+ - u_j^-}, \qquad \boldsymbol{u} \in [0,1]^d, \quad 1 \le k \le l \le n,$$

with $u_j^+ = \min(u_j + h_{k,l}, 1)$, $u_j^- = \max(u_j - h_{k,l}, 0)$ and $\boldsymbol{u}^{j,\pm} = (u_1,\dots,u_j^{\pm},\dots,u_d)$. This estimator is in spirit of section 3 of Kojadinovic et al. (2011).

Note that $\check{\mathbb{D}}^{(\beta)}_{n,m}$ is slightly different to

$$\widetilde{\mathbb{D}}^{(\beta)}_{n,m}(s,\boldsymbol{u}) = \lambda_n(s,1)\check{\mathbb{C}}^{(\beta)}_{n,m}(0,s,\boldsymbol{u}) - \lambda_n(0,s)\check{\mathbb{C}}^{(\beta)}_{n,m}(s,1,\boldsymbol{u}), \qquad (s,\boldsymbol{u}) \in [0,1]^{d+1}.$$

These resample $\widetilde{\mathbb{D}}^{(\beta)}_{n,m}$ can be studied in a future research.

We have the following Proposition (proved in Appendix A.)

Proposition 2.2. *Under the same condition as Proposition 2.1, we have the following result:*

$$\left(\mathbb{D}_{n,m}, \check{\mathbb{D}}^{(1)}_{n,m}, \dots, \check{\mathbb{D}}^{(B)}_{n,m}\right) \rightsquigarrow \left(\mathbb{D}_C, \mathbb{D}_C^{(1)}, \dots, \mathbb{D}_C^{(B)}\right),$$

in $\ell^\infty(\Delta \times [0,1]^d)^{B+1}$*, where for* $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$*, where* $\mathbb{D}_C$ *is defined in* (13) *and* $\mathbb{D}_C^{(1)},\dots,\mathbb{D}_C^{(B)}$ *are independent copies of* $\mathbb{C}_C$*.*

As a corollary of Proposition 3.1 and continuous mapping theorem, we have the following result:

Corollary 2.1. *Let* $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ *be* d*-dimensional independent random vectors with copula* C*, such that for* $b \in (0,1)$ *known and* $m = \lfloor nb\rfloor$*, the random vectors* $\boldsymbol{X}_1,\dots,\boldsymbol{X}_m$ *have c.d.f.* F *and the random vectors* $\boldsymbol{X}_{m+1},\dots,\boldsymbol{X}_n$ *have c.d.f.* F'*.*

Consider the statistic defined in (12) *by*

$$S_{n,m} = \sup_{s\in[0,1]} \int_{[0,1]^d} \{\mathbb{D}_{n,m}(s,\boldsymbol{u})\}^2 \, C_{1:n,m}(\boldsymbol{u}),$$

and re-sampling versions of this statistic defined for $\beta = 1, \ldots, B$ by

$$\check{S}_{n,m}^{(\beta)} = \sup_{s\in[0,1]} \int_{[0,1]^d} \{\check{\mathbb{D}}_{n,m}^{(\beta)}(s, \boldsymbol{u})\}^2 \, \d C_{1:n,m}(\boldsymbol{u}).$$

Under the Condition 2.1,

$$(S_{n,m}, \check{S}_{n,m}^{(1)}, \ldots, \check{S}_{n,m}^{(B)}) \rightsquigarrow (S_C, S_C^{(1)}, \ldots, S_C^{(B)}),$$

where

$$S_C = \sup_{s\in[0,1]} \int_{[0,1]^d} \{\mathbb{D}_C(s, \boldsymbol{u})\}^2 \, \d C(\boldsymbol{u}),$$

and $S_C^{(1)}, \ldots, S_C^{(B)}$ are independent copies of S_C.

An approximate p-value of the test for H_0^m can be obtained by

$$\hat{p}_{n,B}^m = \frac{1}{B} \sum_{\beta=1}^{B} \mathbf{1}\left\{S_{n,m}^{(\beta)} \geq S_{n,m}\right\}.$$

The previous proposition and the Proposition F.1 in the supplementary material of Bücher and Kojadinovic (2015) allow to conclude that the test based on $\hat{p}_{n,m,B}$ will hold its level asymptotically as $n \to \infty$ followed by $B \to \infty$.

3. Break detection in the copula when multiple break times in m.c.d.f.s are known

Suppose in this section that for an integer $p > 0$ and for $j = 1, \ldots, p+1$, the random vectors $\boldsymbol{X}_{m_{j-1}+1}, \ldots, \boldsymbol{X}_{m_j}$ have m.c.d.f.s $F_{1j}, \ldots, F_{dj}$ where $m_0 = 0$, $m_{p+1} = n$ and for $j = 1, \ldots, p$ $m_j = \lfloor nb_j \rfloor$, $0 \leq b_1 < \ldots < b_p \leq 1$ known. In the sequel, $\boldsymbol{m}$ denotes the vector of break points $(m_1, \ldots, m_p)$. Similarly at the section 2, we propose a test for $H_0^{\boldsymbol{m}} = H_{0,c} \cap H_{1,\boldsymbol{m}}$ where $H_{1,\boldsymbol{m}}$ is defined by

$$H_{1,\boldsymbol{m}} : \text{for } j = 1, \ldots, p \ \exists F_{1j}, \ldots, F_{dj} \text{ such that } \boldsymbol{X}_{m_{j-1}+1}, \ldots, \boldsymbol{X}_{m_j} \text{ have m.c.d.f.s } F_{1j}, \ldots, F_{dj}.$$

Here, for $i = 1, \ldots, n$ we consider the random vectors $\hat{\boldsymbol{U}}_{i,\boldsymbol{m}}^{1:n}$ defined by

$$\hat{\boldsymbol{U}}_{i,\boldsymbol{m}}^{1:n} = \begin{cases} (F_{1:m_1,1}(X_{i1}), \ldots, F_{1:m_1,d}(X_{id})) & m_0 + 1 = 1 \leq i \leq m_1 \\ (F_{m_1+1:m_2,1}(X_{i1}), \ldots, F_{m_1+1:m_2,d}(X_{id})) & m_1 < i \leq m_2 \\ \vdots & \vdots \\ \left(F_{m_p+1:n,1}(X_{i1}), \ldots, F_{m_p+1:n,d}(X_{id})\right) & m_p < i \leq m_{p+1} = n, \end{cases}$$

where for $j = 1, \ldots, p+1$ and $q = 1, \ldots, d$, $F_{m_{j-1}+1:m_j,q}$ is the empirical c.d.f. of $X_{m_{j-1}+1q}, \ldots, X_{m_jq}$ defined in (6). For $1 \leq k \leq l \leq n$, denote by $C_{k:l,\boldsymbol{m}}$ the empirical c.d.f. of random vectors $\hat{\boldsymbol{U}}_{1,\boldsymbol{m}}, \ldots, \hat{\boldsymbol{U}}_{n,\boldsymbol{m}}$. More particularly, for $1 \leq k \leq l \leq n$, $1 < m_1 < \ldots < m_p < n$ and $\boldsymbol{u} \in [0,1]^d$, we have

$$C_{k:l,\boldsymbol{m}}(\boldsymbol{u}) = \begin{cases} C_{k:l}(\boldsymbol{u}) & m_1, \ldots, m_p \notin [k, l] \\ \dfrac{1}{l-k+1} \displaystyle\sum_{j=q_1}^{q_2+1} (m'_j - m'_{j-1}) C_{m'_{j-1}+1:m'_j}(\boldsymbol{u}) & \begin{array}{c} m_{q_1}, \ldots, m_{q_2} \in [k, l] \\ {\scriptstyle m'_{q_1-1}=k-1 \text{ and } m'_{q_2+1}=l} \\ {\scriptstyle (m'_{q_1},\ldots,m'_{q_2})=(m_{q_1},\ldots,m_{q_2})} \end{array} \end{cases}.$$

Consider the process $\mathbb{D}_{n,\boldsymbol{m}}$ defined by

$$\mathbb{D}_{n,\boldsymbol{m}}(s, \boldsymbol{u}) = \sqrt{n}\lambda_n(0, s)\lambda_n(s, 1)\{C_{1:\lfloor ns \rfloor,\boldsymbol{m}}(\boldsymbol{u}) - C_{\lfloor ns \rfloor+1:n,\boldsymbol{m}}(\boldsymbol{u})\}, \quad (s, \boldsymbol{u}) \in [0,1]^{d+1}.$$

Proposition 3.1. *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be d-dimensional random vectors with copula C, such that for $0 \leq b_1 \leq \ldots \leq b_p \leq 1$ known and $m_j = \lfloor nb_j \rfloor$ for $j = 1, \ldots, p$, the random vectors $\boldsymbol{X}_{m_{j-1}+1}, \ldots, \boldsymbol{X}_{m_j}$ have m.c.d.f.s $F_{1j}, \ldots, F_{dj}$.*

Then, under Condition 2.1, the process $\mathbb{D}_{n,\boldsymbol{m}}$ converges weakly in $\ell^\infty([0,1]^{d+1})$, to $\mathbb{D}_C$ defined in (13).

The proof of Proposition 3.1 is similar to the the proof of Proposition 2.1 in which the supremas are broken on C_{p+2}^2 supremas.

For $s \in [0,1]$, denote q the integer such that $0 = b_0 \le b_1 < \ldots < b_q < s \le b_{q+1} < \ldots < b_p \le b_{p+1} = 1$. $\mathbb{D}_{n,\boldsymbol{m}}$ rewrites for $(s, \boldsymbol{u}) \in [0,1]^{d+1}$ as

$$\mathbb{D}_{n,\boldsymbol{m}}(s,\boldsymbol{u}) = \lambda_n(s,1)\{\sum_{j=1}^{q} \mathbb{C}_{n,m}(b_{j-1}, b_j, \boldsymbol{u}) + \mathbb{C}_{n,m}(b_q, s, \boldsymbol{u})\} - \lambda_n(0,s)\{\mathbb{C}_{n,m}(s, b_{q+1}, \boldsymbol{u}) + \sum_{j=q+1}^{p} \mathbb{C}_{n,m}(b_j, b_{j+1}, \boldsymbol{u})\}.$$

For $\beta = 1, \ldots, B$, consider the re-sampling versions

$$\check{\mathbb{D}}^{(\beta)}_{n,\boldsymbol{m}}(s,\boldsymbol{u}) = \lambda_n(s,1)\{\sum_{j=1}^{q} \check{\mathbb{C}}^{(\beta)}_{n,m}(b_{j-1}, b_j, \boldsymbol{u}) + \check{\mathbb{C}}^{(\beta)}_{n,m}(b_q, s, \boldsymbol{u})\} - \lambda_n(0,s)\{\check{\mathbb{C}}^{(\beta)}_{n,m}(s, b_{q+1}, \boldsymbol{u}) + \sum_{j=q+1}^{p} \check{\mathbb{C}}^{(\beta)}_{n,m}(b_j, b_{j+1}, \boldsymbol{u})\}.$$

Proposition 3.2. *Under the same condition as Proposition 3.1, the conclusion of proposition 2.2 holds with $\mathbb{D}_{n,\boldsymbol{m}}$ instead of $\mathbb{D}_{n,m}$ and for $\beta = 1, \ldots, B$, $\check{\mathbb{D}}^{(\beta)}_{n,\boldsymbol{m}}$ instead of $\check{\mathbb{D}}^{(\beta)}_{n,m}$.*

This Proposition can be proved in the same way as the prove of Proposition 2.2.

4. Monte Carlo simulations

In all the simulations, d-dimensional observations were considered, with either a Clayton (Cl) copula or a Gumbel–Hougaard (GH) copula defined for $\boldsymbol{u} = (u_1, \ldots, u_d) \in [0,1]^d$ by

$$C^{Cl}_\theta(\boldsymbol{u}) = \max\left(\sum_{j=1}^{d} u_j - d + 1, 0\right)^{-1/\theta}, \quad \theta \ge 1,$$

$$C^{GH}_\theta(\boldsymbol{u}) = \exp\left(-\left[\sum_{j=1}^{d} \{-\log(u_j)\}^\theta\right]^{1/\theta}\right), \quad \theta > 0.$$

In an equivalent way, the Kendall's tau of bivariate margins were specified instead of the parameter θ of the copula. The Monte Carlo experiments were generated using the R Development Core Team (R statistical system 2013) and the *copula* package of Hofert et al. (2015) to sample the Clayton and Gumbel–Hougaard Copulas. The reader may request the corresponding routine by contacting the author.

In a first situation corresponding to the simulations appearing in Table B.1, independent samples of sizes $n = \{50, 100, 200\}$, and dimensions $d = \{2, 3\}$ are considered, where the first $m = \lfloor nb \rfloor$ observations, $b = \{0.1, 0.25, 0.5\}$, have for marginal distributions, normal distributions $N(2,1)$ and for copula a Clayton copula or a Gumbel–Hougaard copula. The bivariate margins have a Kendall's tau of $\tau = \{0.25, 0.5, 0.75\}$. The last $n - \lfloor nb \rfloor$ observations have for marginal distributions, normal distributions $N(0,1)$ and for copula a Clayton copula or Gumbel–Hougaard copula and the bivariate margins have a Kendall's tau of τ. Typically, the samples were generated under $H^m_0 = H_{0,c} \cap H_{1,m}$ where $H_{0,c}$ is defined in (4) and $H_{1,m}$ in (8). The percentage of rejection of H^m_0, based on $S_{n,m}$ defined in (12) with a level $\alpha = 5\%$ were studied.

[Table 1 about here.]

These percentages of rejection are appreciably closed around $\alpha = 5\%$, except for the cases with $b = 0.1$ and $n = 50, 100$ where estimations are calculated on 5 and 10 observations. This may explain the too high percentage of rejection of H^m_0.

In Table B.2 and B.3, independent samples of sizes $n = \{50, 100, 200\}$, and dimensions $d = \{2, 3\}$ are considered where the first $m = \lfloor nb \rfloor$ observations, $b = \{0.1, 0.25, 0.5\}$ have for marginal distributions, normal distributions $N(2,1)$ and for copula a Clayton copula or Gumbel–Hougaard copula. The bivariate margins have a Kendall's tau of $\tau = 0.2$. The last $n - \lfloor nb \rfloor$ observations have for marginal distributions, normal distributions $N(0,1)$ and for copula Clayton copula or Gumbel–Hougaard copula where the bivariate margins have a Kendall's tau of $\tau = \{0.4, 0.6\}$. Typically, the samples were generated under alternative hypotheses $H^m_A = H_{1,c} \cap H_{1,m}$ where $H_{1,m}$ is defined in (8) and with

$$H_{1,c}: \quad \text{There exist } k \in \{1, \ldots, n-1\} \text{ and two copulas } C_1 \text{ and } C_2, \\ \text{such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_k \text{ have copula } C_1 \text{ and } \boldsymbol{X}_{k+1}, \ldots, \boldsymbol{X}_n \text{ have copula } C_2. \quad (16)$$

[Table 2 about here.]

[Table 3 about here.]

The break times $k = \lfloor nt \rfloor$, $t \in \{0.1, 0.25, 0.5\}$ are considered. The percentages of rejection of the hypothesis H_0^m with a level $\alpha = 5\%$ are studied. In the same way, the test for $H_0 = H_{0,c} \cap H_{0,m}$ where $H_{0,m}$ is defined in (3) is considered, based on S_n described in equation (7). The percentages of rejection of H_0^m based on $S_{n,m}$ are closed to the percentages of rejection of H_0 based on S_n. More exactly the percentages of rejection of H_0^m based on $S_{n,m}$ are generally smaller than percentage of rejection of H_0 based S_n for $b = \{0.1, 0.25\}$ and larger for $b = 0.5$ and $t \in \{0.25, 0.5\}$.

Recall that with the hypothesis of a break time m known in the m.c.d.f.s, the rejection of H_0 using S_n does not allow for a conclusion of a break in the copula of observations contrary to the rejection of H_0^m using $S_{n,m}$.

5. Discussions and specific situation

5.1. A Strong mixing condition

Suppose that the random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are drawn from sequences of weakly dependent vectors, in the sense of α-mixing dependence (strong mixing dependence) introduced in Rosenblatt (1956):

Definition 5.1. *Let $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ a sequence of random vectors, and for $a, b \in \bar{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$, denote by $\mathcal{F}_a^b$ the σ-field generated by $(\boldsymbol{X}_i)_{a\leq i\leq b}$. The sequence of α-mixing coefficients $(\alpha_r)_{r\in\mathbb{N}}$ is defined by*

$$\alpha_r = \sup_{t\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^t, B\in\mathcal{F}_{t+r}^{+\infty}} |P(A\cap B) - P(A)P(B)|, \quad r \in \mathbb{N}.$$

The sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ will be said to be strongly mixing as soon as $\alpha_r \underset{r\to+\infty}{\longrightarrow} 0$.

1. The propositions 2.1 and 3.1 remain true if we suppose that the marginal probability integral transforms $\boldsymbol{U}_{1,m}, \ldots, \boldsymbol{U}_{n,m}$ defined in (9) are drawn from a strictly stationary sequence $(\boldsymbol{U}_i)_{i\in\mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(n^{-a})$, $a > 1$. In this case the covariance structure of $\mathbb{Z}_C$ is given by $\mathrm{cov}(\mathbb{Z}_C(s,\boldsymbol{u}), \mathbb{Z}_C(t,\boldsymbol{v})) = \min(s,t)\sum_{k\in\mathbb{Z}} \mathrm{cov}(\mathbf{1}(\boldsymbol{U}_0 \leq \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_k \leq \boldsymbol{v}))$, $(s,\boldsymbol{u})$, $(t,\boldsymbol{v}) \in [0,1]^{d+1}$.
2. The propositions 2.2 and 3.2 remains true if we suppose that the marginal probability integral transforms $\boldsymbol{U}_{1,m}, \ldots, \boldsymbol{U}_{n,m}$ defined in (9) are drawn from a strictly stationary sequence $(\boldsymbol{U}_i)_{i\in\mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(n^{-a})$, $a > 3+3d/2$ and we consider dependent multipliers satisfy (M1)–(M3) appearing in Bücher and Kojadinovic (2015, section 2) with $\ell_n = O(n^{1/2-\gamma})$ for some $0 < \gamma < 1/2$ instead of independent multipliers.

These situations have been studied in Tables B.4 and B.5; sequences of multipliers were simulated using the procedure of (Bücher and Kojadinovic, 2015, The moving average approach, Section 6.1). A standard normal sequence of i.i.d. random variables was used in the construction of multipliers. The value of the bandwidth appearing in the condition (M2) was automatically selected by the procedure described in (Bücher and Kojadinovic, 2015, Section 5) by using the R function *bOptEmpProc* of *npcp* package (Kojadinovic (2014)). The "combining" function ψ appearing in this same procedure was arbitrarily chosen as $\psi = maximum$ (see Politis and White, 2004, Section 4). Finally the function φ appearing in the condition (M3) was the convolution product $\varphi(x) = \kappa_P \star \kappa_P(2x)/\kappa_P \star \kappa_P(0)$, where $\kappa_P = (1 - 6x^2 + 6|x|^3)\mathbf{1}(|x| \leq 1/2) + 2(1-|x|)^3\mathbf{1}(1/2 < |x| \leq 1)$, $x \in \mathbb{R}$.

In Table B.4 and Table B.5, dependent samples of sizes $n = \{100, 200\}$ and $d = \{2, 3\}$ are considered with a break in the variance of marginal distributions at time $m = \lfloor nb \rfloor$, $b = \{0.1, 0.25, 0.5\}$. The samples were generated under $H_0^m = H_{0,c} \cap H_{1,m}$ in Table B.4 and under $(\neg H_{0,c}) \cap H_{1,m}$ in Table B.5 with a break in the copula at time $k = \lfloor nt \rfloor$, $t = \{0.1, 0.25, 0.5\}$. The data are generated from two autoregressive models (AR1) defined by:

$$X_{i+1,j} = 0.5X_{i,j} + \varepsilon_{i+1,j}, \quad j = 1, \ldots, d \text{ and } i \in \mathbb{Z} \quad \text{with } \varepsilon_{ij} \text{ white noise.} \tag{AR1}$$

For $i = 1, \ldots m$ and $j = 1, \ldots, d$ the chosen white noises are $N(0,1)$, and for $i = m+1, \ldots n$ the chosen white noises are $N(0,16)$. For $b \leq t$, the sample is obtained in the following way: let $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_{k+200}$ be a d-variate i.i.d. sample from the copula C_1 and $\boldsymbol{U}_{k+201}, \ldots, \boldsymbol{U}_{n+200}$ be a d-variate i.i.d. sample from the copula C_2 (under H_0^m, $C_1 = C_2$). For $i = 1, \ldots, m+100$, let $\boldsymbol{\varepsilon}_i = (\Phi^{-1}(U_{i1}), \ldots, \Phi^{-1}(U_{id}))$ where Φ is the c.d.f. of the standard normal distribution, and for $i = m+101, \ldots, n+200$, $\boldsymbol{\varepsilon}_i = (4 \times \Phi^{-1}(U_{i1}), \ldots, 4 \times \Phi^{-1}(U_{id}))$. Then, $\boldsymbol{X}_1 = \boldsymbol{\varepsilon}_1$, $\boldsymbol{X}_{m+101} = \boldsymbol{\varepsilon}_{m+101}$ and for $i = 1, \ldots, m+99$, and $i = m+101, \ldots, n+199$ compute recursively

$$\boldsymbol{X}_{i+1} = 0.5\boldsymbol{X}_i + \varepsilon_{i+1}.$$

Finally, we remove the observations $\boldsymbol{X}_1$ to $\boldsymbol{X}_{100}$ and $\boldsymbol{X}_{m+101}$ to $\boldsymbol{X}_{m+200}$. For $b > t$, the sample is obtained in the similar way.

[Table 4 about here.]

[Table 5 about here.]

From $n = 200$, it can seen in Table B.4 that the percentages of rejection of H_0^m are appreciably closed around $\alpha = 5\%$ whereas for the whole of break scenarios in copula (Table B.5), the percentages of rejection of H_0^m are relatively high.

5.2. Specific situation

As an illustration, the bivariate log-returns computed from closing daily quotes of the Dow Jones Industrial Average and the Nasdaq Composite for the years 1987 and 1988 have been studied. This is an interesting situation because the data highlight a change in the m.c.d.f.s at time $m = 202$ (1987-10-19, corresponding to the "Black Monday"). A Cramér–von Mises test (see for example Holmes et al., 2013) can allow to confirm this change. Using the procedure described in Section 2, an approximate p-value of 0,201 was obtained and no evidence against $H_{0,c}$ is reported.

Because a marginal gradual change or a multiple marginal change could lead to a rejection of H_0^m, in the case of rejection of H_0^m, the hypothesis of a unique change in marginal distribution should be confirmed.

5.3. Case of unknown marginal break

It seem interesting not to fix the break time $m = \lfloor nb \rfloor$ and aggregating this term in the best possible way. Nevertheless, such aggregation would be expensive in simulation time, thus the performance of the algorithm should be improved before.

Another interesting way of a future research will be to consider an estimation of the unknown break time instead of m in $S_{n,m}$ and study the associated statistic.

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Appendix A. Proof of Proposition 2.1 and Proposition 2.2

Proof of Proposition 2.1. In the sequel, for $k > l$ the empirical copulas $C_{k:l}$ and $C_{k:l,m}$ are considered as null by convention. Let $b \in (0,1)$ such that $m = \lfloor nb \rfloor$.

Let us consider the two-sided sequential empirical copula process defined in Bücher and Kojadinovic (2015); Bücher et al. (2014) by:

$$\mathbb{C}_n(s,t,\boldsymbol{u}) = \sqrt{n}\lambda_n(s,t)\{C_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})\}, \quad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d, \tag{A.1}$$

where $\Delta = \{(s,t) \in [0,1]^2 | s \leq t\}$. If $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are drawn from a i.i.d. sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ with continuous margins $F_1, \dots, F_d$ then (Proposition 3.3 of Bücher et al. (2014)),

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{C}_n(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_n(s,t,\boldsymbol{u})| \overset{\Pr}{\to} 0, \tag{A.2}$$

with

$$\tilde{\mathbb{C}}_n(s,t,\boldsymbol{u}) = \mathbb{B}_n(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\mathbb{B}_n(s,t,\boldsymbol{u}^{\{j\}}), \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d, \tag{A.3}$$

and

$$\mathbb{B}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d, \tag{A.4}$$

where the vector $\boldsymbol{U}_i$, $i = 1, \dots, n$, are the vectors of the probability integral tranforms $\boldsymbol{U}_i = (F_1(X_{i1}), \dots, F_d(X_{id}))$.

Here, we only suppose that $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ have same copula C.
Define the process $(s,t,\boldsymbol{u}) \mapsto \tilde{\mathbb{C}}_{n,m}(s,t,\boldsymbol{u})$ similarly to $(s,t,\boldsymbol{u}) \mapsto \tilde{\mathbb{C}}_n(s,t,\boldsymbol{u})$ in (A.3) and (A.4) using the vectors $\boldsymbol{U}_{\lfloor ns \rfloor+1,m}, \dots, \boldsymbol{U}_{\lfloor nt \rfloor,m}$ instead of $\boldsymbol{U}_{\lfloor ns \rfloor+1}, \dots, \boldsymbol{U}_{\lfloor nt \rfloor}$ in (A.4).

The Proposition 2.1 can be seen as a corollary of the following Lemma:

Lemma A1. *Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be d-dimensional independent random vectors with copula C, such that for a fixed integer $m = \lfloor nb \rfloor$, $b \in (0,1)$ known, the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_m$ have m.c.d.f.s $F_1, \dots, F_d$ and the random vectors $\boldsymbol{X}_{m+1}, \dots, \boldsymbol{X}_n$ have m.c.d.f.s $F'_1, \dots, F'_d$. Under Condition 2.1,*

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{C}_{n,m}(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_{n,m}(s,t,\boldsymbol{u})| \overset{\Pr}{\to} 0.$$

consequently,

$$\mathbb{C}_{n,m} \rightsquigarrow \mathbb{C}_C \qquad \text{in } \ell^\infty(\Delta \times [0,1]^d).$$

Proof of Lemma A1. We will demonstrated the 3 following convergences:

$$\sup_{(s,t,\boldsymbol{u})\in(\Delta\cap[0,b]^2)\times[0,1]^d} |\mathbb{C}_{n,m}(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_{n,m}(s,t,\boldsymbol{u})| \overset{\Pr}{\to} 0, \tag{A.5}$$

$$\sup_{(s,t,\boldsymbol{u})\in(\Delta\cap[b,1]^2)\times[0,1]^d} |\mathbb{C}_{n,m}(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_{n,m}(s,t,\boldsymbol{u})| \overset{\Pr}{\to} 0, \tag{A.6}$$

$$\sup_{(s,t,\boldsymbol{u})\in[0,b]\times[b,1]\times[0,1]^d} |\mathbb{C}_{n,m}(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_{n,m}(s,t,\boldsymbol{u})| \overset{\Pr}{\to} 0. \tag{A.7}$$

On $(\Delta \cap [0,b]^2) \times [0,1]^d$ or $(\Delta \cap [b,1]^2) \times [0,1]^d$, we have $\mathbb{C}_{n,m}(s,t,\boldsymbol{u}) = \mathbb{C}_n(s,t,\boldsymbol{u})$ and $\tilde{\mathbb{C}}_{n,m}(s,t,\boldsymbol{u}) = \tilde{\mathbb{C}}_n(s,t,\boldsymbol{u})$, hence the desirated convergence follow from (A.2):

$$\begin{aligned}
&\sup_{(s,t,\boldsymbol{u})\in(\Delta\cap[0,b]^2)\times[0,1]^d} |\mathbb{C}_{n,m}(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_{n,m}(s,t,\boldsymbol{u})| \\
&= \sup_{(s,t,\boldsymbol{u})\in(\Delta\cap[0,b]^2)\times[0,1]^d} |\mathbb{C}_n(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_n(s,t,\boldsymbol{u})| \\
&= \sup_{(s,t,\boldsymbol{u})\in(\Delta\cap[0,b]^2)\times[0,1]^d} |\mathbb{C}^*_n(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}^*_n(s,t,\boldsymbol{u})| \\
&\leq \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{C}^*_n(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}^*_n(s,t,\boldsymbol{u})| \overset{\Pr}{\to} 0,
\end{aligned} \tag{A.8}$$

where $\mathbb{C}_n^*$ and $\tilde{\mathbb{C}}_n^*$ are the processes $\mathbb{C}_n$ and $\tilde{\mathbb{C}}_n$ constructed from $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{\lfloor nb \rfloor}, \boldsymbol{X}^*_{\lfloor nb \rfloor+1}, \ldots, \boldsymbol{X}^*_n$ with $\boldsymbol{X}^*_{\lfloor nb \rfloor+1}, \ldots, \boldsymbol{X}^*_n$ such as $\boldsymbol{X}_1, \ldots, \boldsymbol{X}^*_n$ are i.i.d.

Because by construction, for all $(s,t,\boldsymbol{u}) \in [0,b] \times [b,1] \times [0,1]^d$ we have $\mathbb{C}_{n,m}(s,t,\boldsymbol{u}) = \mathbb{C}_n(s,b,\boldsymbol{u}) + \mathbb{C}_n(b,t,\boldsymbol{u})$ and $\tilde{\mathbb{C}}_{n,m}(s,t,\boldsymbol{u}) = \tilde{\mathbb{C}}_n(s,b,\boldsymbol{u}) + \tilde{\mathbb{C}}_n(b,t,\boldsymbol{u})$, the suprema in (A.7) is bounded by

$$\sup_{(s,\boldsymbol{u})\in[0,b]\times[0,1]^d} |\mathbb{C}_n(s,b,\boldsymbol{u}) - \tilde{\mathbb{C}}_n(s,b,\boldsymbol{u})| + \sup_{(t,\boldsymbol{u})\in[b,1]\times[0,1]^d} |\mathbb{C}_n(b,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_n(b,t,\boldsymbol{u})|$$

using the triangle inequality, hence the same argumentation as (A.8) allows us to conclude.

Using the fact that for any $m \in \{1, \ldots, n-1\}$, the random vectors $\boldsymbol{U}_{1,m}, \ldots, \boldsymbol{U}_{n,m}$ are i.i.d., we obtain the weak limit of $(s,t,\boldsymbol{u}) \mapsto \mathbb{B}_{n,m}(s,t,\boldsymbol{u})$ in $\ell^\infty(\Delta \times [0,1]^d)$ to $(s,t,\boldsymbol{u}) \mapsto \mathbb{Z}_C(t,\boldsymbol{u}) - \mathbb{Z}_C(s,\boldsymbol{u})$ (see for example the Theorem 2.12.1 of van der Vaart and Wellner, 2000) hence the process $(s,\boldsymbol{u}) \mapsto \mathbb{D}_{n,m}(s,\boldsymbol{u})$ converges weakly in $\ell^\infty([0,1]^{d+1})$ to $\mathbb{C}_C^0(s,\boldsymbol{u}) - s\mathbb{C}_C^0(1,\boldsymbol{u})$.

Proof of Proposition 2.2. Proceeding as the proof of Proposition 4.3 in Bücher et al. (2014), for $\beta \in \{1, \ldots, B\}$ and $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$, put

$$\mathbb{B}_{n,m}^{(\beta)}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \xi_i^{(\beta)} \{\mathbf{1}(\boldsymbol{U}_{i,m} \le \boldsymbol{u}) - C(\boldsymbol{u})\},$$

$$\mathbb{C}_{n,m}^{(\beta)}(s,t,\boldsymbol{u}) = \mathbb{B}_{n,m}^{(\beta)}(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u}) \mathbb{B}_{n,m}^{(\beta)}(s,t,\boldsymbol{u}^{\{j\}}).$$

and the similar versions $\mathbb{B}_n^{(\beta)}$ and $\mathbb{C}_n^{(\beta)}$ based on $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ instead of $\boldsymbol{U}_{1,m}, \ldots, \boldsymbol{U}_{n,m}$. From Lemma A1, the fact that $\boldsymbol{U}_{1,m}, \ldots, \boldsymbol{U}_{n,m}$ are i.i.d. and using the Theorem 2.1 in Bücher and Kojadinovic (2015) and the continuous mapping theorem, we have that

$$\left(\mathbb{C}_{n,m}, \mathbb{C}_{n,m}^{(1)}, \ldots, \mathbb{C}_{n,m}^{(B)}\right) \rightsquigarrow \left(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \ldots, \mathbb{C}_C^{(B)}\right)$$

in $\ell^\infty(\Delta \times [0,1]^d)^{B+1}$. With this result, it is sufficient to demonstrate the following convergences: for $\beta = 1, \ldots, B$,

$$\sup_{(s,\boldsymbol{u})\in[0,b]\times[0,1]^d} |\check{\mathbb{C}}_{n,m}^{(\beta)}(s,b,\boldsymbol{u}) - \mathbb{C}_{n,m}^{(\beta)}(s,b,\boldsymbol{u})| \overset{\Pr}{\to} 0,$$

$$\sup_{(s,\boldsymbol{u})\in[b,1]\times[0,1]^d} |\check{\mathbb{C}}_{n,m}^{(\beta)}(b,s,\boldsymbol{u}) - \mathbb{C}_{n,m}^{(\beta)}(b,s,\boldsymbol{u})| \overset{\Pr}{\to} 0,$$

$$\sup_{(s,\boldsymbol{u})\in[0,b]\times[0,1]^d} |\check{\mathbb{C}}_{n,m}^{(\beta)}(0,s,\boldsymbol{u}) - \mathbb{C}_{n,m}^{(\beta)}(0,s,\boldsymbol{u})| \overset{\Pr}{\to} 0, \qquad \sup_{(s,\boldsymbol{u})\in[b,1]\times[0,1]^d} |\check{\mathbb{C}}_{n,m}^{(\beta)}(s,1,\boldsymbol{u}) - \mathbb{C}_{n,m}^{(\beta)}(s,1,\boldsymbol{u})| \overset{\Pr}{\to} 0,$$

$$\sup_{\boldsymbol{u}\in[0,1]^d} |\check{\mathbb{C}}_{n,m}^{(\beta)}(0,b,\boldsymbol{u}) - \mathbb{C}_{n,m}^{(\beta)}(0,b,\boldsymbol{u})| \overset{\Pr}{\to} 0, \qquad \sup_{\boldsymbol{u}\in[0,1]^d} |\check{\mathbb{C}}_{n,m}^{(\beta)}(b,1,\boldsymbol{u}) - \mathbb{C}_{n,m}^{(\beta)}(b,1,\boldsymbol{u})| \overset{\Pr}{\to} 0.$$

In fact, using the same argumentation as (A.8) and term (B.2) appearing in Bücher et al. (2014), the previous convergences are automatically verified.

Appendix B. Simulation study

Appendix: simulation study

Table B.1: Percentage of rejection of H_0^m based on the $S_{n,m}$ statistic, computed from 1000 random samples of size $n = \{50, 100, 200\}$ generated under H_0^m, where C is either a d-dimensional Clayton (Cl) or a d-dimensional Gumbel-Hougaard (GH) copula with Kendall's tau of τ. The $\lfloor nb \rfloor$ first observations have for marginal distributions, normal distributions $N(2,1)$ and the $n - \lfloor nb \rfloor$ last observations have for marginal distributions, normal distributions $N(0,1)$. The test is based on S_n^m and replications are computed using independent multipliers

			Cl			GH		
n	d	τ	$b = 0.1$	$b = 0.25$	$b = 0.5$	$b = 0.1$	$b = 0.25$	$b = 0.5$
50	2	0.25	11.3	9.5	7.4	7.9	6.9	7.0
		0.50	16.3	10.4	4.8	10.4	7.5	4.0
		0.75	36.1	11.3	5.4	18.2	5.8	1.7
	3	0.25	9.3	9.3	8.1	5.3	6.3	5.1
		0.50	10.8	9.3	7.0	3.9	3.0	2.5
		0.75	3.4	2.1	1.7	1.0	1.3	0.6
100	2	0.25	8.2	7.6	6.6	4.3	2.9	3.8
		0.50	8.0	7.0	4.9	6.6	5.0	4.4
		0.75	11.5	6.0	2.1	5.8	4.1	1.5
	3	0.25	5.7	6.0	6.3	5.4	5.5	5.1
		0.50	8.1	8.5	7.9	3.6	3.2	3.4
		0.75	3.8	2.7	1.1	1.0	1.2	0.3
200	2	0.25	4.7	5.2	4.5	4.9	5.1	5.2
		0.50	4.3	5.5	4.2	5.3	5.1	4.4
		0.75	5.1	4.0	1.8	4.5	3.1	1.1
	3	0.25	5.2	5.6	4.8	4.0	4.1	4.2
		0.50	7.0	6.7	5.5	3.7	3.7	3.1
		0.75	3.6	3.2	2.8	2.2	2.0	1.1

Table B.2: Percentage of rejection of H_0^m computed from 1000 random samples of size $n = \{50, 100, 200\}$ generated under $H_A = H_{1,c} \cap H_{1,m}$ defined in (16) and (8), where the first $\lfloor nt \rfloor$ observations, $t \in \{0.1, 0.25, 05\}$ have for copula a d-dimensional Clayton (Cl) copula with Kendall's tau of 0.2 and the last $n - \lfloor nt \rfloor$ have for copula a d-dimensional Clayton copula (Cl) with Kendall's tau of τ. The first $m = \lfloor nb \rfloor$ observations, $b \in \{0.1, 0.25, 0.5\}$ have for marginal distributions, normal distributions $N(0, 1)$ and the $n - m$ last observations have for marginal distributions, normal distributions $N(2, 1)$. Two different tests (based on S_n and S_n^m) are compared and replications are computed using independent multipliers

n	d	t	τ	$S_{n,m}$ $b = 0.1$	S_n $b = 0.1$	$S_{n,m}$ $b = 0.25$	S_n $b = 0.25$	$S_{n,m}$ $b = 0.5$	S_n $b = 0.5$
50	2	0.10	0.4	15.9	10.5	10.5	15.1	7.7	6.7
			0.6	39.9	28.5	23.0	45.0	10.7	8.4
		0.25	0.4	21.6	19.3	18.0	21.1	11.6	9.6
			0.6	56.6	48.5	48.5	60.0	31.8	19.9
		0.50	0.4	23.8	23.8	22.0	25.9	17.0	15.3
			0.6	59.0	56.3	55.7	63.7	46.3	43.2
	3	0.10	0.4	12.8	12.6	11.3	15.9	8.5	6.0
			0.6	26.0	25.4	20.7	40.9	13.6	8.7
		0.25	0.4	20.0	21.0	21.7	25.0	14.8	10.5
			0.6	57.8	62.6	56.2	67.6	47.2	30.3
		0.50	0.4	28.7	31.2	27.5	35.6	24.9	24.6
			0.6	75.9	79.4	76.9	82.3	73.5	72.3
100	2	0.10	0.4	10.5	17.5	9.5	27.4	6.4	8.1
			0.6	35.4	47.9	28.4	72.8	19.5	25.4
		0.25	0.4	23.8	33.1	24.0	40.6	17.3	13.6
			0.6	78.0	84.2	74.7	90.1	62.7	47.8
		0.50	0.4	32.4	38.3	32.4	45.5	29.4	30.0
			0.6	84.0	89.7	85.1	95.2	81.9	85.0
	3	0.10	0.4	11.8	16.3	11.5	22.9	9.7	7.9
			0.6	36.6	49.4	33.1	69.0	25.7	21.8
		0.25	0.4	30.8	40.0	31.1	45.1	26.9	19.7
			0.6	86.7	91.9	87.0	94.4	82.4	63.6
		0.50	0.4	43.1	53.0	42.4	59.1	41.9	41.8
			0.6	95.8	97.0	95.4	98.5	95.6	96.1
200	2	0.10	0.4	11.2	30.2	11.7	45.8	10.2	16.6
			0.6	49.0	81.4	42.5	96.1	37.7	70.1
		0.25	0.4	35.6	56.3	36.2	68.7	31.1	31.7
			0.6	93.4	98.0	93.5	99.7	91.1	89.1
		0.50	0.4	50.3	65.6	49.8	76.0	49.3	58.4
			0.6	99.1	99.9	98.9	100.0	99.0	99.5
	3	0.10	0.4	12.7	29.2	12.6	44.1	11.2	14.3
			0.6	61.5	81.6	59.7	94.4	57.9	59.8
		0.25	0.4	46.2	66.5	48.8	72.7	45.0	36.7
			0.6	98.7	99.9	99.2	99.9	99.0	94.6
		0.50	0.4	69.0	82.3	69.3	88.8	70.2	75.2
			0.6	100.0	100.0	100.0	100.0	100.0	100.0

Table B.3: Percentage of rejection of H_0^m computed from 1000 random samples of size $n = \{50, 100, 200\}$ generated under $H_A = H_{1,c} \cap H_{1,m}$ defined in (16) and (8), where the first $\lfloor nt \rfloor$ observations, $t \in \{0.1, 0.25, 05\}$ have for copula a d-dimensional Gumbel–Hougaard (GH) copula with Kendall's tau of 0.2 and the last $n - \lfloor nt \rfloor$ have for copula a d-dimensional Gumbel–Hougaard (GH) with Kendall's tau of τ. The first $m = \lfloor nb \rfloor$ observations, $b \in \{0.1, 0.25, 0.5\}$ have for marginal distributions, normal distributions $N(0, 1)$ and the $n - m$ last observations have for marginal distributions, normal distributions $N(2, 1)$. Two different tests (based on S_n and S_n^m) are compared and replications are computed using independent multipliers

n	d	t	τ	$S_{n,m}$ $b = 0.1$	S_n $b = 0.1$	$S_{n,m}$ $b = 0.25$	S_n $b = 0.25$	$S_{n,m}$ $b = 0.5$	S_n $b = 0.5$
50	2	0.10	0.4	11.2	9.0	8.5	13.7	5.6	4.8
			0.6	23.5	22.6	15.1	31.4	8.9	6.5
		0.25	0.4	18.2	14.9	15.2	19.5	11.1	7.3
			0.6	45.9	41.8	39.4	49.1	28.8	15.8
		0.50	0.4	20.7	17.9	19.9	22.0	17.1	15.7
			0.6	52.9	55.0	49.5	60.1	46.5	43.3
	3	0.10	0.4	7.1	8.2	6.8	12.2	6.1	5.8
			0.6	13.4	16.3	10.3	26.1	8.2	4.7
		0.25	0.4	12.2	15.8	14.0	19.0	11.5	8.5
			0.6	41.3	49.6	41.6	54.5	36.1	22.9
		0.50	0.4	20.9	25.9	21.5	27.9	21.8	18.9
			0.6	66.1	72.1	68.0	73.8	63.9	62.4
100	2	0.10	0.4	9.1	11.5	7.9	21.6	6.6	6.6
			0.6	26.5	44.0	18.3	59.5	14.1	17.2
		0.25	0.4	19.4	28.3	20.5	33.1	17.1	11.6
			0.6	66.7	76.0	62.4	84.2	55.9	39.3
		0.50	0.4	26.7	34.6	27.2	41.5	24.8	25.8
			0.6	81.6	87.5	80.2	91.0	79.3	81.2
	3	0.10	0.4	6.9	11.9	6.6	18.0	6.2	5.3
			0.6	25.1	38.8	21.8	55.1	19.5	12.6
		0.25	0.4	21.5	29.4	22.8	34.0	21.2	14.5
			0.6	79.1	87.6	79.6	90.0	76.8	55.1
		0.50	0.4	39.3	50.8	38.5	55.1	39.3	39.7
			0.6	93.2	95.9	93.4	96.9	92.8	91.9
200	2	0.10	0.4	8.7	26.4	8.2	42.4	7.8	15.7
			0.6	40.5	79.3	35.2	92.7	33.4	56.2
		0.25	0.4	31.3	52.1	29.2	62.6	28.3	26.2
			0.6	91.1	98.3	91.5	99.8	90.3	80.4
		0.50	0.4	46.3	63.0	45.2	72.6	44.2	51.7
			0.6	99.1	99.8	99.0	100.0	99.4	99.6
	3	0.10	0.4	8.2	25.3	9.9	34.9	9.6	9.5
			0.6	52.5	80.4	48.6	89.4	48.6	40.0
		0.25	0.4	40.2	62.2	42.0	64.0	42.0	31.0
			0.6	98.0	99.4	98.1	99.6	97.9	91.1
		0.50	0.4	65.9	79.5	65.5	85.5	66.1	69.7
			0.6	99.9	100.0	100.0	100.0	100.0	100.0

Table B.4: Percentage of rejection of H_0^m computed from 1000 samples of size $n = \{100, 200\}$ generated under H_0^m and from two (AR1) models, with d-dimensional Clayton (Cl) or Gumbel–Hougaard (GH) stationary copula with Kendall's tau of τ. The $m = \lfloor nb \rfloor$ first observations, $b \in \{0.1, 0.25, 0.5\}$ have stationary margins $N(0,1)$ and the $n - \lfloor nb \rfloor$ last observations have stationary margins $N(0,16)$. The test is based on S_n^m and replications are computed using dependent multipliers

			Cl			GH		
n	d	τ	$b = 0.1$	$b = 0.25$	$b = 0.5$	$b = 0.1$	$b = 0.25$	$b = 0.5$
100	2	0.25	9.0	10.6	9.6	8.3	10.3	8.3
		0.50	11.2	8.2	8.5	11.5	8.1	6.2
		0.75	12.2	11.2	5.1	12.9	8.1	5.0
	3	0.25	9.7	9.9	10.6	7.0	10.3	8.8
		0.50	9.9	10.7	8.1	5.0	6.8	7.2
		0.75	4.5	3.4	3.3	2.5	2.8	1.4
200	2	0.25	5.9	8.0	6.5	7.6	6.5	5.6
		0.50	6.1	7.1	5.1	3.6	5.4	4.8
		0.75	4.3	4.6	3.4	3.4	2.9	0.7
	3	0.25	6.2	4.8	6.4	5.0	6.3	8.2
		0.50	5.2	5.7	4.5	4.9	4.3	4.6
		0.75	2.1	3.6	1.9	1.0	1.1	0.6

Table B.5: Percentage of rejection of H_0^m computed from 1000 samples of size $n = \{100, 200\}$ generated under $\neg H_{0,c} \cap H_{1,m}$ and from two (AR1) models, where the first $\lfloor nt \rfloor$ observations, $t \in \{0.1, 0.25, 05\}$ have for stationary copula a d-dimensional Clayton (Cl) copula (resp. Gumbel–Hougaard Copula) with Kendall's tau of 0.2 and the last $n - \lfloor nt \rfloor$ have for stationary copula a bidimensional Clayton copula (Cl) (resp. Gumbel–Hougaard Copula) with Kendall's tau of τ. The $\lfloor nb \rfloor$ first observations have m.c.d.f. $N(0,1)$ and the $n - \lfloor nb \rfloor$ last observations have m.c.d.f. $N(0,16)$. The test is based on S_n^m and replications are computed using dependent multipliers

				Cl			GH		
n	d	t	τ	$b = 0.1$	$b = 0.25$	$b = 0.5$	$b = 0.1$	$b = 0.25$	$b = 0.5$
100	2	0.10	0.4	14.8	13.4	10.5	12.7	12.3	9.1
			0.6	36.5	23.9	10.7	33.1	19.6	9.4
		0.25	0.4	24.6	25.4	17.5	22.8	20.4	15.4
			0.6	62.5	62.8	45.9	62.0	55.7	45.1
		0.50	0.4	31.9	32.2	25.2	27.0	24.8	25.8
			0.6	74.7	71.0	69.1	73.8	72.1	66.4
	3	0.10	0.4	12.2	13.2	11.8	11.3	10.6	9.0
			0.6	34.9	23.1	13.8	26.0	18.3	11.1
		0.25	0.4	23.7	31.5	20.9	23.7	25.3	18.7
			0.6	74.0	77.0	64.9	67.3	72.6	59.0
		0.50	0.4	38.8	40.8	36.1	32.3	35.0	35.5
			0.6	85.1	87.6	86.8	85.6	86.6	84.8
200	2	0.10	0.4	10.8	10.4	9.6	10.5	8.6	7.9
			0.6	39.9	21.9	11.3	36.3	20.1	12.2
		0.25	0.4	27.9	29.5	22.1	24.2	26.6	20.6
			0.6	80.9	81.5	72.1	79.8	82.0	71.9
		0.50	0.4	39.5	37.5	35.7	35.7	38.4	32.6
			0.6	91.9	92.5	90.3	92.5	90.9	92.0
	3	0.10	0.4	14.9	11.4	8.9	10.4	9.6	8.4
			0.6	45.8	26.6	17.6	45.2	21.4	15.6
		0.25	0.4	37.3	38.9	31.9	30.6	35.0	29.1
			0.6	93.5	94.1	86.8	90.7	93.6	86.9
		0.50	0.4	54.6	53.0	54.0	51.5	51.8	51.9
			0.6	99.3	98.0	98.0	98.5	98.9	98.4